Seat No. : _______________

DO-130

December-2017

M.Sc., Sem.-I

403 : Mathematics

Complex Analysis – I (Old)

Time : 3 Hours] **[Max. Marks : 70**

1. (a) Exhibit all the roots as vertices of certain regular polygon geometrically and also pointing out the principal root for $8^{1/6}$. **7**

OR

(a) What does $|z - 5| = |z - 5i|$ represent ? Justify your answer in two different ways.

(b) Answer any **two** of the following briefly : **4**

(i) Find Arg $(\sqrt{3} - i)^6$.

(ii) Write $(1 + \sqrt{3}i)^{-10}$ in the rectangular form $x = iy$.

(iii) Show that $|\operatorname{Re} z| + |\operatorname{Im} z| \leq \sqrt{2}\,|z|$.

(c) Answer **all** of the following very briefly : **3**

(i) Sketch the set $|z - 2 + i| \leq 1$ and determine if it is a domain.

(ii) Sketch the set $|2z + 3| > 4$ and determine if it is a domain.

(iii) Find Arg(πi).

2. (a) Show by discussing in detail the appropriate example that mere satisfaction of C-R equations is not sufficient for differentiability of f at a point. **7**

OR

(a) Suppose $f : D \rightarrow \mathbb{C}$ satisfies $f'(z) = 0$ for all $z \in D$. Show that $f(z)$ is constant on D. By giving an appropriate example show that the condition that D is a domain can not be dropped.

(b) Answer any **two** of the following briefly : **4**

(i) Discuss differentiability of $f(z) = x^2 + iy^2$. And find the derivative at points where ever it is differentiable.

(ii) Discuss differentiability of $f(z) = z - \overline{z}$.

(iii) Show that $\lim\limits_{z \to \infty} f(z) = \infty$ iff $\lim\limits_{z \to 0} \dfrac{1}{f\left(\frac{1}{z}\right)} = 0$. Using this show that

$$\lim_{z \to \infty} \frac{2z^3 - 1}{z^2 + 1} = \infty.$$

(c) Answer **all** of the following very briefly : **3**

(i) What do you mean by a neighbourhood of ∞ ?

(ii) When do you say that f is analytic at z_0 ?

(iii) What is the reflection property ?

3. (a) When do you say that ν is a Harmonic Conjugate of u ? Show that $f(z) = u(x, y) + i\nu(x, y)$ is analytic in a domain D if and only if ν is a Harmonic Conjugate of u. Find a Harmonic Conjugate of $u(x, y) = x^2 - y^2$. **7**

OR

(a) How is Log(z) defined on $D = \mathbb{C} - \{0\}$? Discuss the points where it is discontinuous. Justify your claims.

(b) Answer any **two** of the following briefly : **4**

(i) Solve the equation $\sin z = 2$ for z.

(ii) Show that $\sin^{-1}(-i) = n\pi + i(-1)^{n+1} \ln(1 + \sqrt{2}), n \in \mathbb{Z}$.

(iii) Find the image of the line $x = 0$ under the map $f(z) = \exp(z) = e^z$. What is it in the w-plane ?

(c) Answer **all** of the following very briefly : **3**

(i) Find all complex numbers z such that $e^z = 1$.

(ii) Find Log $(-e\mathrm{i})$.

(iii) Find all the roots of the equation $\log z = i\pi/2$.

4. (a) Suppose f is analytic throughout the closed region R consisting of all points interior to and on a simple closed contour C and f' is continuous there. Then show that $\int_C f(z)\,dz = 0$ **7**

OR

(a) Suppose that f is continuous on a domain D. Show that f has anti-derivative F in D if the integrals of $f(z)$ around closed contours lying entirely in D all have value zero.

(b) Answer any **two** of the following briefly : **4**

(i) Show that $\left| \int_C (e^z - \bar{z})dz \right| \leq 60$ where C is the boundary of the triangle with vertices at the points 0, 3i, and –4 oriented in counterclockwise directions.

(ii) Evaluate $\int_{|z|=\pi} \frac{1}{z}\,dz$, from the definition of contour integral.

(iii) Evaluate $\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz$.

(c) Answer **all** of the following very briefly : **3**

(i) What is meant by a simple closed contour ? Explain giving example.

(ii) Giving the meanings of all the notations, establish $\left| \int_C f(z)dz \right| \leq M\,L$.

(iii) Find the contour integral $\int_{|z|=1} \sin z\,dz$. Justify your answer.

5. (a) Derive Cauchy Integral Formula. **7**

OR

(a) Derive the main part of the proof of the Extension of Cauchy Integral Formula.

(b) Answer any **two** of the following briefly : **4**

(i) Evaluate the integral $\displaystyle\int_{|z|=1} \frac{\exp(2z)}{z^4}\, dz$.

(ii) Suppose C is positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate the integral $\displaystyle\int_C \frac{\cos z}{z(z^2+8)}\, dz$.

(iii) Evaluate the integral $\displaystyle\int_{|z-i|=2} \frac{dz}{(z^2+4)^2}$.

(c) Answer **all** of the following very briefly : **3**

(i) Evaluate the integral $\displaystyle\int_{|z-i|=2} \frac{dz}{z^2+4}$.

(ii) Find $\displaystyle\int_{|z-i|=2} (z^4 + z^3 + z^2 + z + 1)\, dz$.

(iii) Find $\displaystyle\int_C \frac{z}{2z+1}\, dz$, where C is positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$.

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1. (a) When do you say that $f : \mathbb{C} \to \mathbb{C}$ is continuous at point z_0 ? Show that if $f : \mathbb{C} \to \mathbb{C}$ is continuous and non-zero at z_0, it is also non-zero throughout a neighbourhood of z_0. **7**

OR

(a) Prove that $\left| \dfrac{z - w}{1 - \bar{z}w} \right| = 1$ if either $|z| = 1$ or $|w| = 1$.

(b) When do you say that the complex number z_0 is an accumulation point of a subset S of complex numbers ? Show that a finite set of points $z_1, z_2, \ldots z_n$ cannot have any accumulation point. **7**

OR

(I) Show that $\sqrt{2}\,|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$.

(II) If c is any n^{th} root of unity other than 1, show that $1 + c + c^2 \ldots + c^{n-1} = 0$.

2. (a) What are the C-R equations ? Show that C-R equations are satisfied at $z = z_0$ when f is differentiable at $z = z_0$. Discuss the differentiability of $f(z) = 2xy + i(x^2 - y^2)$. **7**

OR

(a) Suppose that a function f and its conjugate $\bar{f}$ both are analytic on a domain D. Show that f is constant on D. Derive from this that if f is analytic on D having constant modulus on D then also f is constant on D.

(b) Suppose $f : D \to \mathbb{C}$ satisfies $f'(z) = 0$ for all $z \in D$. Show that $f(z)$ is constant on D. By giving an appropriate example show that the condition that D is a domain cannot be dropped. **7**

OR

(b) Suppose D is a domain which contains a segment of the x-axis and is symmetric about that axis and $f : D \to \mathbb{C}$ is analytic. Show that f has reflection property on D if and only if $f(x)$ is real for each point x on the segment. Does $f(z) = \sin z$ satisfy the reflection property ? Why ? Justify using the above Reflection Principle.

3. (a) How is Log (z) defined on $D = C - \{0\}$? Discuss the points where it is discontinuous. Justify your claims. On what restricted domain will Log (z) be differentiable ? What is its derivative ? How ? **7**

OR

(a) Find the integral $\int_C f(z)\,dz$, where $f(z) = \frac{z+2}{z}$ and C is the upper-half circle of radius 2 around origin in the counter clock-wise direction.

(b) Suppose that f is continuous on a domain D. Show that f has anti-derivative F in D if and only if the integrals of $f(z)$ around closed contours lying entirely in D all have value zero. **7**

OR

(b) Find all the roots in $\mathbb{C}$ of the equation $\cos z = 2$.

4. (a) Suppose f is analytic throughout the closed region R consisting of all points interior to and on a simple closed contour C and f' is continuous there. Then show that $\int_C f(z)\, dz = 0$. **7**

OR

(a) Derive Cauchy Integral Formula.

(b) State and prove Liouville's theorem. What does this theorem say for the entire function that is not a constant function ? What can you conclude for the function cos z ? **7**

OR

(b) Suppose $f(z)$ is analytic and $|f(z)| \leq |f(z_0)|$ on $|z - z_0| < \in$. Show that f is constant throughout the neighbourhood.

5. Attempt any **seven** from the following : **14**

(1) Find square roots of 2i.

(2) Find the Arg (π) and Arg (πi).

(3) Find the Arg ($-\pi - \pi$i).

(4) Discuss the differentiability of the function $f(z) = |z|^2$.

(5) Find the values of Log(1 – i) and Log (– ei).

(6) Find the principal values of i^i and $(-i)^i$.

(7) Find the value of the integral $\int_C \frac{dz}{z^2 + 4}$ where C is the circle $|z - i| = 2$ oriented in the positive direction.

(8) Evaluate the integral $\int\limits_{|z|=1} \frac{\exp(2z)}{z^4}\,dz$.

(9) Suppose $f : \{z \in \mathbb{C} : |z| = 1\} \to \mathbb{C}$ is defined as $f(z) = z$. What are the maximum and minimum values of $|f(z)|$? Where are these values attained ? Are they all boundary points of D ?